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General class of BPS saturated dyonic black holes as exact superstring solutions

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Abstract

We show that a four-parameter generating solution for a general class of four-dimensional, spherically-symmetric, static, dyonic BPS saturated solutions of leading-order effective equations of toroidally compactified heterotic string theory is an exact string solution. The corresponding ten-dimensional background defines a conformal sigma-model which is a particular case of a ‘chiral null model’ with curved ‘transverse’ part. The exact conformal invariance is a consequence of the chiral null structure of the ‘electric’ part of the model and the $N = 4$ world-sheet supersymmetry of its transverse ‘magnetic’ part. The sigma-model action has a remarkable covariance under both target space T -duality and the electro-magnetic S -duality transformations, and it illustrates the relationship between string-string duality in six dimensions and S -duality in four dimensions. In general, there exists a large class of exact six-dimensional superstring solutions described by chiral null models with four-dimensional transverse parts represented by $N = 4$ supersymmetric σ -models with metrics conformal to hyper-Kähler ones.

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1. *Introduction.* Recent recognition [1, 2] that Bogomol'nyi-Prasad-Sommerfield (BPS) saturated states should play an important role in the full non-perturbative dynamics of string theory, has triggered a renewed interest in supersymmetric solutions of different effective field theories in various dimensions (for a review, see [3] and references therein). The study of such configurations may, in particular, shed light on duality symmetry between certain strongly coupled and weakly coupled string vacua.

These backgrounds have minimal energy in their class, i.e., they saturate the Bogomol'nyi bound on the energy, and are usually obtained as extrema of the leading-order effective string-theory (supergravity) actions. These solutions admit Killing spinors and thus preserve some of the supersymmetries.

To interpret such a configuration as a solution of *string theory* (and not just of supergravity theory) and thus to achieve an adequate understanding of its properties and its role in the context of string theory, one should identify the corresponding conformal two-dimensional (2d) field theory. The first step in this direction is to find a conformal 2d σ -model with couplings $(G_{\mu\nu}, B_{\mu\nu}, \Phi, \dots)$ which reduce to the background fields in the leading order in α' , i.e., which extend the leading-order background to an exact string solution.

There are large classes of conformal σ -models, or exact string solutions, which actually retain their leading-order form: they are not modified by α' -corrections provided one uses a special scheme (for a review, see [4] and references therein). Such are 'chiral null models' [5], which, in particular, describe (after the Kaluza-Klein truncation) supersymmetric extreme *electric* four-dimensional black-hole-type solutions (see, e.g., [6, 5, 7, 8, 9, 10]). The special chiral null structure of the corresponding σ -model implies that these solutions are exact not only in the type II superstring or heterotic string theory, but also in the bosonic string theory [5]. On the other hand, it is the extended supersymmetry that is crucial for the exactness of the extreme *magnetic* black hole solutions, i.e., they are exact only as heterotic or type II superstring solutions [11, 12, 13, 14, 15].

The aim of the present paper is to show that a class of four-dimensional spherically-symmetric static *dyonic* BPS saturated solutions of leading-order effective equations of toroidally compactified heterotic string theory (for a review, see [16] and references therein) is actually a class of *exact string solutions* by presenting the corresponding conformal σ -model. By embedding these solutions into ten-dimensional string theory we shall find that the resulting superstring action is a particular case of the generalized chiral null model [17, 5, 4] with *curved* four-dimensional 'transverse' part. Although the general background with the non-vanishing electric and magnetic charges corresponds to a model with only $N = 1$ space-time supersymmetry, so that the resulting σ -model possesses only $N = 2$ world-sheet supersymmetry, the chiral null structure of the 'light-cone' ('electric') part of the σ -model action, along with the extended $N = 4$ world-sheet supersymmetry³ of the four-dimensional 'transverse' ('magnetic')

³In what follows $N = 4$ world-sheet supersymmetry would mean $(4, 0)$ supersymmetry in the heterotic string case and $(4, 1)$ supersymmetry in the type II (or 'embedded' heterotic) case.

part, will suffice to argue that the full 2d theory is conformal to all orders. We shall thus reach a remarkable conclusion that the electric and magnetic charges associated with two different compact Kaluza-Klein dimensions can be superposed at the level of conformal σ -model without need to introduce extra terms in the 2d action. This is a non-trivial result, given that the ‘dyonic’ σ -model does not factorise into the sum of ‘electric’ and ‘magnetic’ parts since both depend on the same radial coordinate.

2. General Dyonic BPS Saturated Black Hole Solution. The supersymmetric backgrounds we are going to consider are solutions of the four-dimensional $N = 4$ supersymmetric effective action of the heterotic string compactified on a six-torus (see, e.g., [18] and references therein). At generic points of the moduli space of toroidally compactified heterotic string the set of BPS saturated, supersymmetric, static, spherically symmetric configurations is both $O(6, 22, Z)$ and $SL(2, Z)$ duality invariant. In [19] the explicit form of a general class of such configuration with 56 charges subject to one constraint was given. It corresponds to the states which can be obtained by acting with $SL(2, Z)$ transformations on configurations with *zero axion and most general allowed dyonic charges*.

The latter set of solutions are $O(6, 22, Z)$ orbits of dyonic configurations whose ‘left’ as well as ‘right’ electric and magnetic charges are *orthogonal*, i.e., light-like in the $O(6, 22, Z)$ sense. It turns out [19] that the generating solution for this set is the one which has the non-trivial scalar fields being represented only by the diagonal components of the internal metric of the six-torus (g_{mm} , $m = 1, \dots, 6$) and the four-dimensional dilaton (ϕ_4), whose asymptotic values may be set equal to one and zero, respectively.

The explicit form of this *generating solution* depends on four parameters: two magnetic and two electric charges. The magnetic and electric charges are associated with two *different* internal dimensions, i.e., two different $U(1)$ groups. The magnetic (electric) charges correspond to the Kaluza-Klein vector field $A_\varphi^{(1)m}$ ($A_t^{(1)n}$) and the vector field originating from the antisymmetric tensor components $A_{\varphi m}^{(2)}$ ($A_{tn}^{(2)}$). Here the indices t and φ refer to the time and polar angle coordinates of four-dimensional space-time and $m \neq n$ are a pair of the six-torus indices (for the explicit form of the solution in terms of the four-dimensional fields see [19]). Without loss of generality, the non-zero charges can be chosen to be $P_1^{(1)}, P_1^{(2)}, Q_2^{(1)}, Q_2^{(2)}$, i.e., to be the charges of the gauge fields associated with the first two circles of the six-torus.⁴

The generating solution depends only on the four ‘screened’ charges ($\mathbf{P}_1^{(1)}, \mathbf{P}_1^{(2)}, \mathbf{Q}_2^{(1)}, \mathbf{Q}_2^{(2)} \equiv (\eta_P P_1^{(1)}, \eta_P P_1^{(2)}, \eta_Q Q_2^{(1)}, \eta_Q Q_2^{(2)})$). Here $\eta_{P,Q} = \pm 1$ are parameters in the Killing spinor constraints which are chosen so that $\eta_P \text{sign}(P_1^{(1)} + P_1^{(2)}) = 1$ and $\eta_Q \text{sign}(Q_2^{(1)} + Q_2^{(2)}) = 1$, thus yielding a non-negative ADM mass $M_{BPS} = \mathbf{P}_1^{(1)} + \mathbf{P}_1^{(2)} + \mathbf{Q}_2^{(1)} + \mathbf{Q}_2^{(2)}$ for the four-dimensional configuration.

⁴ The upper index refers to the origin (Kaluza-Klein metric or two-form) of the corresponding $U(1)$ gauge field and the lower index indicates the number of the internal circular dimension.

The four-dimensional space-time structure depends on the relative signs of the two magnetic and the two electric charges. When the relative signs of the two magnetic and the two electric charges are the same, we refer to such solutions as *regular*. They have a Reissner-Nordström-type horizon, null singularity, or naked singularity when all four, only three (or two), or only one of the charges are nonzero, respectively. On the other hand, when the relative signs for the two magnetic (and/or two electric) charges are opposite, we refer to such solutions as *singular*; they always have a naked singularity and repel massive particles [9, 20]. In addition, the latter set of configurations become massless [8, 9, 20] when the two screened electric (and two magnetic) charges have opposite signs and equal magnitudes, and may contribute to the enhancement of local gauge symmetry [21, 20] as well as local supersymmetry [20] at special points of moduli space.

The solution of [19] was expressed in terms of the fields appearing in the four-dimensional, low-energy effective action. In terms of the corresponding fields of the ten-dimensional heterotic string theory (see, e.g., [18]) $G_{\mu\nu}$, $B_{\mu\nu}$, Φ , A_μ^I ($\mu, \nu = 0, \dots, 9$, $I = 1, \dots, 16$), i.e., the string-frame metric, the two-form field, the dilaton and the Abelian (Cartan subalgebra) Yang-Mills fields, the solution takes the form:

$$ds_{10}^2 = \sum_{n=3}^6 dy_n^2 + F(r) [2dt dy_2 + K(r) dy_2^2] \quad (1)$$

$$+ f(r) \left\{ k(r) [dy_1 + \mathbf{P}_1^{(1)}(1 - \cos \theta) d\varphi]^2 + k^{-1}(r) [dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2)] \right\},$$

$$B_{\varphi,1} = \mathbf{P}_1^{(2)}(1 - \cos \theta), \quad B_{t,2} = F(r), \quad \Phi = \frac{1}{2} \ln [F(r)f(r)], \quad A_\mu^I = 0, \quad (2)$$

where

$$F^{-1} = 1 + \frac{\mathbf{Q}_2^{(2)}}{r}, \quad K = 1 + \frac{\mathbf{Q}_2^{(1)}}{r}, \quad f = 1 + \frac{\mathbf{P}_1^{(2)}}{r}, \quad k^{-1} = 1 + \frac{\mathbf{P}_1^{(1)}}{r}. \quad (3)$$

Here t and (r, θ, φ) are time and the spherical coordinates of the four-dimensional space-time, while (y_1, \dots, y_6) are the periodic coordinates of the six-torus. The radial coordinate r is chosen in such a way that the horizon (or singularity) of the four-dimensional Einstein-frame metric for the regular solutions is at $r = 0$, while the singular solutions have a naked singularity at $r = \max[\min(|\mathbf{P}_1^{(1)}|, |\mathbf{P}_1^{(2)}|), \min(|\mathbf{Q}_2^{(1)}|, |\mathbf{Q}_2^{(2)}|)]$. Note that the non-trivial $(t, r, \theta, \varphi; y_1, y_2)$ -part of the solution is six-dimensional. These solutions admit a straightforward multi-center generalization (see also below).

It turns out [19] that the non-zero magnetic and non-zero electric charges each break $\frac{1}{2}$ of the maximal number of supersymmetries. Thus, purely electric (or purely magnetic) configurations preserve $\frac{1}{2}$, while dyonic solutions preserve only $\frac{1}{4}$ of the $N = 4$ supersymmetry in four dimensions.⁵ Consequently, the corresponding full

⁵The first and the second sets of configurations belong to the vector- and hyper- supermultiplets with highest spins 1 and $\frac{3}{2}$ [22, 23], respectively.

(heterotic) string σ -model action will have only $N = 2$ world-sheet supersymmetry. However, its transverse ‘magnetic’ part will have an increased, $N = 4$, world-sheet supersymmetry.

3. Conformal Chiral Null Models with $N = 4$ Supersymmetric Transverse Part.

Our goal is to write down the world-sheet supersymmetric heterotic or type II superstring σ -model action corresponding to the background (1)–(3). The world-sheet action will turn out to correspond to a special case of a generalized chiral null model which is conformally invariant to all orders in α' , in spite of the fact that the full σ -model action will have only $N = 2$ world-sheet supersymmetry. A novel mechanism of preserving conformal invariance which will be at work here is based on combining the chiral null structure of the ‘electric’ part of the model with the extended $N = 4$ world-sheet supersymmetry of its four-dimensional transverse ‘magnetic’ part. Let us first describe a general class of such conformal chiral null models.

The chiral null models [5, 17, 4] is a class of 2d σ -models which generalize both plane wave type and fundamental string type models

$$L = F(x)\partial u \left[\bar{\partial}v + K(u, x)\bar{\partial}u + \mathcal{A}_i(u, x)\bar{\partial}x^i \right] + (G_{ij} + B_{ij})(x) \partial x^i \bar{\partial}x^j + \mathcal{R}\Phi(u, x). \quad (4)$$

Here u, v are ‘light-cone’ coordinates, x^i are ‘transverse space’ coordinates and $\mathcal{R} \equiv \frac{1}{4}\alpha'\sqrt{g^{(2)}}R^{(2)}$. The affine symmetry $v' = v + h(z)$ implies the existence of a conserved chiral null current. The corresponding metric has a null Killing vector and the generalized connection with torsion has a special holonomy.

A remarkable property of the model (4) is that there exists a renormalisation scheme in which it is conformal to all orders in α' provided (i) the ‘transverse’ σ -model $(G_{ij} + B_{ij})\partial x^i \bar{\partial}x^j$ is conformal when supplemented with a dilaton coupling $\phi(x)$ (so that (G_{ij}, B_{ij}, ϕ) represent an exact string solution), and (ii) the functions $F^{-1}, K, \mathcal{A}_i, \Phi$ satisfy⁶

$$F^2(-\omega F^{-1} + \partial^i \phi \partial_i F^{-1}) = 0, \quad F(-\omega K + \partial^i \phi \partial_i K) = 0, \quad \omega = \frac{1}{2}\nabla^2 + O(\alpha'),$$

$$F(-\frac{1}{2}\hat{\nabla}_i \mathcal{F}^{ij} + \partial_i \phi \mathcal{F}^{ij}) = 0, \quad \mathcal{F}_{ij} \equiv \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i, \quad \Phi = \phi + \frac{1}{2} \ln F. \quad (5)$$

Here ω is the scalar anomalous dimension operator. In general, it contains (G_{ij}, B_{ij}) -dependent corrections to all orders in α' (see [24, 4]).

The simplest example of the chiral null model is the one with the *flat* transverse space $G_{ij} + B_{ij} = \delta_{ij}$ and constant (or linear) dilaton ϕ_0 . At the points where F is non-vanishing the conditions (5) then reduce to

$$\partial^i \partial_i F^{-1} = 0, \quad \partial^i \partial_i K = 0, \quad \partial_i \mathcal{F}^{ij} = 0, \quad \Phi = \phi_0 + \frac{1}{2} \ln F. \quad (6)$$

This model describes a large class of exact string solutions, in particular, higher dimensional plane waves, fundamental strings and, upon dimension reduction, four-dimensional supersymmetric electric black hole-type solutions [5, 4, 7, 8, 10]. The

⁶In what follows we shall consider for simplicity the special case when K, \mathcal{A}_i, Φ are u -independent. Then the action is also invariant under $v' = v - \eta(x)$, $\mathcal{A}'_i = \mathcal{A}_i + \partial_i \eta$.

electric backgrounds are embedded into a higher dimensional model by assuming that a linear combination of u and v is a compact Kaluza-Klein dimension y , i.e., that the electric field originates from the ‘light-cone’ (u, v) part of the model, while the transverse spatial part remains flat. On the other hand, in order to embed the *magnetic* field, one has to consider models with non-trivial transverse spatial part. This suggests that the *dyonic* solutions discussed above should be described by a more general chiral null model (4) with a *curved* transverse part.

It is indeed possible to obtain exact solutions with curved transverse space when the conformal field theory corresponding to the transverse part of (4) is known explicitly [17, 4].⁷ Here we shall consider another important exactly solvable case, when the transverse σ -model has *extended* $N = 4$ world-sheet supersymmetry. Then one can argue, following [25, 26, 27], that there exists a scheme in which not only the transverse σ -model couplings (G_{ij}, B_{ij}, ϕ) , but also the ‘tachyonic’ operator ω and thus the functions $F, K, \mathcal{A}_i, \Phi$ retain their leading-order form. In what follows we shall set \mathcal{A}_i -couplings in (4) equal to zero. As a result, (6) is replaced by

$$\partial_i(e^{-2\phi}\sqrt{G}G^{ij}\partial_j F^{-1}) = 0, \quad \partial_i(e^{-2\phi}\sqrt{G}G^{ij}\partial_j K) = 0, \quad \Phi = \phi(x) + \frac{1}{2}\ln F(x). \quad (7)$$

As explained in [27], a $(4, 0)$ supersymmetric σ -model is exactly conformal in a special scheme where the $(4, 0)$ supersymmetry is preserved. It is thus sufficient to demand the $(4, 0)$ supersymmetry of the transverse σ -model in order to ensure that the transverse σ -model gives rise to an exact (‘left-right asymmetric’) solution of the heterotic string theory. The same background will be represented by $(4, 1)$ supersymmetric σ -model in the context of type II superstring theory (in some special cases the $(4, 1)$ σ -model may have $(4, 4)$ supersymmetry [11]). The corresponding exact type II superstring solution can be reinterpreted also as another, ‘symmetric’, heterotic string solution which is obtained by embedding the generalized Lorentz connection into the gauge group (see, e.g., [11, 15]).⁸

The conditions for a heterotic (type II) σ -model to admit a $(4, 0)$ ($(4, 1)$) supersymmetry were discussed in [25, 28, 26, 11, 29]. Below we shall consider a particular model with a *four*-dimensional transverse part such that its metric is

$$G_{ij} = f(x)g_{ij}, \quad \nabla^2 f = 0, \quad (8)$$

⁷In that case the ‘tachyonic’ operator ω is given by the zero-mode part of the CFT Hamiltonian H . Fixing a particular scheme (e.g., the one where H has the standard ‘Klein-Gordon with a dilaton’ form), one is able to determine the exact form of the background fields (G_{ij}, B_{ij}, ϕ) and F, K, \mathcal{A}_i . This produces, in particular, the exact solutions which are ‘hybrids’ of the gauged Wess-Zumino-Witten (WZW) and plane wave (or fundamental string) solutions [17].

⁸The resulting gauge field background is of the next to the leading order in α' (and therefore can be ignored in solving the effective field equations), i.e., the ‘asymmetric’ and ‘symmetric’ heterotic string solutions agree to the leading order in α' . The schemes in which these two exact heterotic string solutions are exact are related by local covariant field redefinitions involving terms of all orders in α' [27, 5].

where g is a hyper-Kähler metric and $\nabla \equiv \nabla(g)$ [11].⁹ Then the torsion and dilaton should satisfy $H^{ijk} = -G^{-1/2} \epsilon^{ijkn} \partial_n \ln f$ and $\phi = \frac{1}{2} \ln f$. We shall assume that the four-dimensional hyper-Kähler metric g_{ij} has at least one translational isometry (in x_4 -direction). Then this metric can be put into the form $(x_i = (x_s, x_4), s = 1, 2, 3)$

$$g_{ij} dx^i dx^j = k(x) [dx_4 + a_s(x) dx^s]^2 + k^{-1}(x) dx_s dx^s, \quad (9)$$

where k and a_s depend on x_s only, and satisfy the conditions

$$\partial_s \partial^s k^{-1} = 0, \quad \partial_p a_q - \partial_q a_p = \epsilon_{pqs} \partial^s k^{-1}, \quad (10)$$

which imply the self-duality of the curvature of g_{ij} (here ϵ_{pqs} is the flat-space antisymmetric tensor and the indices are contracted using δ_{pq}). Depending on the asymptotic behaviour of k^{-1} these are (multi-center) Eguchi-Hanson or Taub-NUT gravitational instantons (see, e.g., [30]). Note that for g_{ij} in (9) the condition on $f(x)$ in (8) becomes simply the ‘flat’ one, $\partial_i \partial^i f = 0$.¹⁰

We shall thus consider the (1,0) (or (1,1)) supersymmetric¹¹ six-dimensional chiral null model with (4,0) (or (4,1)) supersymmetric¹² ‘transverse’ part which has the following bosonic term in the Lagrangian

$$L = F(x) \partial u [\bar{\partial} v + K(x) \bar{\partial} u] + \frac{1}{2} \mathcal{R} \ln F(x) + L_{\perp}, \quad (11)$$

$$\begin{aligned} L_{\perp} = f(x) & \left[k(x) (\partial x_4 + a_s(x) \partial x^s) (\bar{\partial} x_4 + a_s(x) \bar{\partial} x^s) + k^{-1}(x) \partial x_s \bar{\partial} x^s \right] \\ & + b_s(x) (\partial x_4 \bar{\partial} x^s - \bar{\partial} x_4 \partial x^s) + \mathcal{R} \phi(x), \end{aligned} \quad (12)$$

where, in addition to (10), we shall assume that $f = f(x_s)$ and

$$\partial_s \partial^s f = 0, \quad \partial_p b_q - \partial_q b_p = \epsilon_{pqs} \partial^s f, \quad \phi = \frac{1}{2} \ln f. \quad (13)$$

Choosing F and K to be independent of x_4 and observing that the transverse metric $G_{\perp} \equiv (G_{ij})$ and the dilaton in (12),(13) are such that (for *any* f and k) $e^{-2\phi} \sqrt{G_{\perp}} G_{\perp}^{pq} = \delta^{pq}$ ($p, q = 1, 2, 3$), we conclude that the conditions (7) on F and K take the *flat space* form (cf. (6))

$$\partial_s \partial^s F^{-1} = 0, \quad \partial_s \partial^s K = 0. \quad (14)$$

Their solutions are thus given by harmonic functions of x_s which are *independent* of a particular choice of the functions f, k, a_s, b_s in L_{\perp} . The above conditions (10),(13),(14)

⁹Examples of other four-dimensional (4,0) models with torsion which have metrics not conformal to hyper-Kähler ones were considered in [29].

¹⁰While k and a_s are assumed to depend on x_s only, f may, in general, depend also on x_4 , thus allowing for a more general set of solutions (see below).

¹¹The full heterotic (type II) model is actually (2,0) ((2,1)) supersymmetric [5], in agreement with the $N = 1$ four-dimensional space-time supersymmetry of the corresponding background.

¹²When g_{ij} is flat ($k = 1$) the corresponding (4,1) type II (or ‘symmetric’ heterotic) ‘transverse’ σ -model is actually (4,4) supersymmetric [11]. The same may be true also in the general case of (9),(10).

are just the leading-order (one-loop) conformal invariance conditions for the bosonic σ -model (11). However, there exists a scheme in which they are *exact* (all-loop) conditions for the conformal invariance of the corresponding (1,0) (or (1,1)) supersymmetric σ -model.

To summarize, the (1,1) supersymmetric model (11),(12) represents an exact solution of the type II superstring theory. There are also two heterotic string solutions associated with the σ -model (11),(12). The ‘asymmetric’ one is obtained by considering its (1,0) supersymmetric extension and arguing that the chiral null structure combined with the (4,0) supersymmetry of the ‘transverse’ part is sufficient for its exact conformal invariance in a certain scheme.¹³ The ‘symmetric’ heterotic string solution is obtained by embedding the ‘*transverse*’ (‘magnetic’) part of the Lorentz connection ω_- into the gauge group, i.e., by adding the gauge field background to make the transverse σ -model anomaly-free and (4,1) supersymmetric as in the special cases considered in [11, 15]. Note that the *full* six-dimensional heterotic σ -model (11),(12) remains only (2,0) supersymmetric since the embedding into the gauge group is not possible for the ‘electric’ part of the Lorentz connection which has a non-compact holonomy [5] (such an embedding is not actually needed for the exact conformal invariance [5]).

If the harmonic functions F, K, f, k are chosen so that they approach constant values at large x (as in the case of (3)), i.e., that at large x the six-dimensional σ -model (11),(12) becomes a free theory with constant dilaton, then the central charge c of this conformal σ -model has the free-theory value (c is constant and hence can be evaluated at large x).

Note also that the model (11),(12) is covariant under the separate σ -model duality (or target space, T -duality) transformations in the two isometric coordinates $u \rightarrow \tilde{u}$ [5] and $x_4 \rightarrow \tilde{x}_4$ combined with¹⁴

$$F \rightarrow K^{-1}, \quad K \rightarrow F^{-1}, \quad f \rightarrow k^{-1}, \quad k \rightarrow f^{-1}, \quad a_s \rightarrow b_s, \quad b_s \rightarrow a_s. \quad (15)$$

In the case of the simple harmonic functions in (3) this transformation is equivalent to $\mathbf{Q}_2^{(1)} \leftrightarrow \mathbf{Q}_2^{(2)}$ and $\mathbf{P}_1^{(1)} \leftrightarrow \mathbf{P}_1^{(2)}$.

4. *Dyonic BPS Backgrounds as Exact Superstring Solutions.* Finally, we are able to relate the above class of chiral null models to the general class of four-dimensional dyonic BPS saturated states described by the backgrounds (1)–(3). The non-trivial

¹³ A subtlety that appears here is related to the presence of the σ -model anomaly implying the modification $H \rightarrow \hat{H}$ of the torsion $H = dB$ by the Lorentz Chern-Simons term in the expressions for the conformal anomaly β -functions. This modification is trivial in the purely ‘electric’ case $f = k = 1$ [5] (where $d\hat{H} = 2\alpha' \text{tr}(R \wedge R) = 0$), but is present in the ‘magnetic’ case [11, 15]. Then one is to assume that there exists a deformation of B_{ij} by higher order α' -terms which solves the conformal invariance equations. Because of the underlying (4,0) supersymmetry of the transverse model it should be possible to ‘undo’ this deformation by a local change of a scheme [26, 27].

¹⁴We are assuming that a_s and b_s are parallel as in the magnetic case discussed below. In general, the duality transformation in x_4 produces an extra torsion term $(b_p a_q - a_p b_q) \partial x^p \bar{\partial} x^q$ in the action.

six-dimensional part of these backgrounds corresponds to a special case of the σ -model (11),(12) where F^{-1}, K, f, k^{-1} are four spherically symmetric one-center solutions (3) of the three-dimensional Laplace equation, and

$$u = y_2, \quad v = 2t, \quad x_4 = y_1, \quad dx_s^2 = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad r^2 = x_s x^s, \\ a_s dx^s = \mathbf{P}_1^{(1)}(1 - \cos \theta) d\varphi, \quad b_s dx^s = \mathbf{P}_1^{(2)}(1 - \cos \theta) d\varphi. \quad (16)$$

Thus, the model (11)–(13),(10) promotes the backgrounds (1)–(3) to *exact superstring solutions*.

One advantage of having identified the conformal σ -model behind the leading-order solution (1)–(3) is that now various generalizations become obvious. For example, the multi-center, or ‘rotating’, or ‘Taub-NUT’-type backgrounds are obtained by choosing more general expressions for the harmonic functions F^{-1}, K, f, k^{-1} . The model (11),(12) admits also an important extension which is found by assuming that f may depend also on x_4 and by replacing the b_s -terms by $B_{ij}(x) \partial x^i \bar{\partial} x^j$ with the generalized torsion $\hat{H}_{ijk} = -2\sqrt{G} G^{nl} \epsilon_{ijkn} \partial_l \phi$. Another possibility would be to use a generic $N = 4$ supersymmetric model as the transverse part, relaxing the condition of isometry in x_4 direction on the hyper-Kähler metric g_{ij} in (8). These generalizations yield new types of genuine six-dimensional solutions which are worth further investigation. The special case of $F = K = k = 1$ and $O(4)$ -symmetric f corresponds to the five-brane solution [11]. One can also consider solutions periodic in x_4 (analog of periodic instantons [12]) which at distances large compared to the period of x_4 correspond to four-dimensional backgrounds.

For regular solutions with both $\mathbf{Q}_2^{(2)}$ and $\mathbf{P}_1^{(1)}$ non-vanishing we can consider the ‘throat limit’ $r \rightarrow 0$ of the conformal model (11),(12),(16),(3).¹⁵ It then takes the following simple form

$$L_{r \rightarrow 0} = \left(p \partial z \bar{\partial} z + q \partial \tilde{y}_2 \bar{\partial} \tilde{y}_2 + 2e^{-z} \partial \tilde{y}_2 \bar{\partial} t \right) \\ + p \left[\partial \tilde{y}_1 \bar{\partial} \tilde{y}_1 + \partial \varphi \bar{\partial} \varphi + \partial \theta \bar{\partial} \theta - 2 \cos \theta \partial \tilde{y}_1 \bar{\partial} \varphi \right], \quad (17)$$

where we have introduced the notation: $z = -\ln r$, $z \rightarrow \infty$ and $\tilde{y}_1 = y_1/\mathbf{P}_1^{(1)} + \varphi$, $\tilde{y}_2 = y_2/\mathbf{Q}_2^{(2)}$, $p = \mathbf{P}_1^{(1)} \mathbf{P}_1^{(2)}$, $q = \mathbf{Q}_2^{(1)} \mathbf{Q}_2^{(2)}$. It is easy to show that (up to the issue of periodicities of coordinates) the terms in the first bracket are equivalent to the Lagrangian of the $SL(2, R)$ WZW model (see, e.g., Appendix B of [5]) while the terms in the second bracket represent the Lagrangian of the $SU(2)$ WZW model (see, e.g., [31, 14]). Thus the model becomes equivalent to the direct product of the $SL(2, R)$ and $SU(2)$ WZW theories divided by discrete subgroups.¹⁶ The levels k

¹⁵In this case the dilaton $\Phi = \frac{1}{2} \ln(Ff)$ approaches a constant for $r \rightarrow 0$. Other cases are related to this one by T -duality (15).

¹⁶See also [32] for related observations. Similar ‘throat-limit’ conformal models were discussed in [33].

of the two WZW models are both equal to $4p$. The central charge of this superconformal theory is then $c = [3(k+2)/k + 3/2] + [3(k-2)/k + 3/2] = 6 + 3$, i.e., it is the same as that of the theory corresponding to a flat background.

Some special cases of the solution (11),(12),(16),(3) obtained by choosing special values of the parameters $(\mathbf{P}_1^{(1)}, \mathbf{P}_1^{(2)}, \mathbf{Q}_2^{(1)}, \mathbf{Q}_2^{(2)})$ are:

- *Zero magnetic charges:* For $\mathbf{P}_1^{(1,2)} = 0$ ($f = k = 1$) the model (11) is the chiral null model with *flat* transverse part discussed in detail in [5, 4] (see also [8, 34]). The case with $\mathbf{Q}_2^{(2)} = 0$ ($F = 1$) corresponding, from the four-dimensional point of view, to the extreme electric black hole solution in Kaluza-Klein theory, is described by the five-dimensional plane wave with flat transverse part [35]. The case with $\mathbf{Q}_2^{(1)} = 0$ ($K = 1$) is related to the one with $\mathbf{Q}_2^{(2)} = 0$ by the duality in y_2 -direction (see (15)) and describes equivalent four-dimensional background. The self-dual case $\mathbf{Q}_2^{(1)} = \mathbf{Q}_2^{(2)}$ ($K = F^{-1}$) corresponding to the extreme electric $a = 1$ dilatonic black hole [36], is represented by the five-dimensional fundamental string solution [6].¹⁷
- *Zero electric charges:* For $\mathbf{Q}_1^{(1,2)} = 0$ ($F = K = 1$) the background (1)–(3) is a generalized five-brane-type solution not studied before. The internal space of the five-brane (9) is a hybrid of the gravitational instanton and the H -instanton. The case with $\mathbf{P}_1^{(1)} = 0$ ($k = 1$) corresponds to the $O(3)$ symmetric analog of the five-brane [38, 11] whose four-dimensional manifestation is the H -monopole [39, 12, 40]. In the case with $\mathbf{P}_1^{(2)} = 0$ ($f = 1$) related to the one with $\mathbf{P}_1^{(1)} = 0$ by the duality in y_1 -direction (see (15)), the space internal to the five-brane describes the Taub-NUT gravitational instanton (with the charge $\mathbf{P}_1^{(1)}$ being quantized in terms of the radius of y_2 to ensure the regularity of the solution). Its four-dimensional ‘image’ is the Kaluza-Klein monopole [41, 13]. The self-dual case $\mathbf{P}_1^{(1)} = \mathbf{P}_1^{(2)}$ ($f = k^{-1}$) corresponds [14, 15] to the extreme magnetic $a = 1$ dilatonic black hole in four dimensions [36].
- *Dyonic case:* The choice $\mathbf{Q}_2^{(2)} = 0$, $\mathbf{P}_1^{(2)} = 0$ ($F = f = 1$) corresponds to the general supersymmetric Kaluza-Klein black hole solution [42], which is described by the six-dimensional Brinkmann pp-wave background with curved transverse space [43].

5. *Duality Transformations.* We would like to end with a discussion of S -duality at the level of the six-dimensional conformal σ -model defined by (11)–(13),(10),(3) and its relation to string-string duality. Since the electro-magnetic S -duality is a non-perturbative, four-dimensional symmetry, which maps (see, e.g., [18]) one solution

¹⁷These four-dimensional electric solutions have the same type of charge assignments as certain elementary string excitations and may thus be identified with them (see, e.g., [37] and references therein).

of the leading-order four-dimensional low-energy effective heterotic string equations into the dual one, the string σ -models corresponding to the two S -dual solutions are not expected to be related in any simple way. The above conformal σ -model is a counter-example to this expectation. Applying the S -duality transformation to the four-dimensional dyonic solution (1)–(3) one gets the S -dual dyonic solution with the two electric and two magnetic charges associated with the internal dimensions y_1 and y_2 , respectively. This S -dual solution can then be elevated back to the effectively six-dimensional solution (1)–(3), with the related σ -model being again (11)–(13), but now with y_1 and y_2 *interchanged*. The transformation $y_1 \leftrightarrow y_2$ and $\mathbf{Q}_n^{(1,2)} \leftrightarrow \mathbf{P}_n^{(1,2)}$, or, more generally, $u \leftrightarrow x_4$ and

$$F \rightarrow f^{-1}, \quad K \rightarrow k^{-1}, \quad f \rightarrow F^{-1}, \quad k \rightarrow K^{-1}, \quad (18)$$

is thus the counterpart of the four-dimensional S -duality at the level of the six-dimensional conformal σ -model.¹⁸

The *six*-dimensional model (11)–(13) provides a remarkable illustration of a relationship between heterotic-type II string-string duality in *six* dimensions and S -duality of heterotic string compactified on six-torus (T^6) [44, 2]. Indeed, the leading terms of the effective six-dimensional action of the heterotic string compactified on T^4 and type II string compactified on $K3$ are related by the following (‘string-string duality’) transformation of the metric $G_{\mu\nu}$, the two-form field $B_{\mu\nu}$ and the dilaton Φ [2]: $G' = e^{-2\Phi}G$, $dB' = e^{-2\Phi} * dB$, $\Phi' = -\Phi$. Applied to the background fields of the six-dimensional conformal σ -model (11)–(13), representing a solution in both dual theories, this transformation maps the σ -model into *itself* with

$$F \rightarrow f^{-1}, \quad K \rightarrow K, \quad f \rightarrow F^{-1}, \quad k \rightarrow k, \quad (19)$$

and a_s and b_s defined by (10),(13).¹⁹

Repeating the sequence of string-string duality transformation (19) and T -duality transformation (15) twice yields (18) which is the four-dimensional electro-magnetic S -duality transformation. This result is in agreement with the observation [2] that the toroidal $SL(2)_T$ duality of type II string compactified on $K3 \times T^2$ implies the $SL(2)_S$ duality of heterotic string compactified on T^6 .

Interestingly, the self-dual case of the σ -model (11),(12), corresponding to the fixed point $F = K^{-1} = f^{-1} = k$ (and $a_s = b_s$) of the duality transformations

¹⁸Note that the four-dimensional dilaton corresponding to (1)–(3) which is defined by $e^{-2\phi_4} = e^{-2\Phi}(G_{y_1 y_1} G_{y_2 y_2})^{1/2} = (KF^{-1}kf^{-1})^{1/2}$ is invariant under the T -duality (15), but indeed changes sign under the S -duality $(F, K) \leftrightarrow (f^{-1}, k^{-1})$, i.e., $\mathbf{Q}_n^{(1,2)} \leftrightarrow \mathbf{P}_n^{(1,2)}$. The same applies to the six-dimensional dilaton $\Phi = \frac{1}{2} \ln(Ff)$ (see also below).

¹⁹When applied to more general σ -models with x_4 -dependent functions mentioned above, the six-dimensional string-string duality (19) relates, in particular, the six-dimensional fundamental string solution [45] ($K = k = 1$, $f = 1$, $F^{-1} = 1 + a/x^2$) and the six-dimensional image ($F = 1$, $K = k = 1$, $f = 1 + a/x^2$) of the five-brane solution [11] (see [46]).

(15),(18),(19), describes in four dimensions [19] the extreme Reissner-Nordström solution. The corresponding σ -model has a particularly simple structure:²⁰

$$L = \partial y_1 \bar{\partial} y_1 + \partial y_2 \bar{\partial} y_2 + 2a_s(x) \partial y_1 \bar{\partial} x^s + 2F(x) \partial y_2 \bar{\partial} t + F^{-2}(x) \partial x_s \bar{\partial} x^s, \quad (20)$$

where we have used the notation in (16), i.e., $y_1 = x_4$, $y_2 = u$, $t = \frac{1}{2}v$. The electro-magnetic duality corresponds to interchanging the third and fourth terms and $y_1 \leftrightarrow y_2$. This model is a special case of

$$L = \partial y_n \bar{\partial} y^n + 2A_\mu^n(x) \partial y_n \bar{\partial} x^\mu + g_{pq}(x) \partial x^p \bar{\partial} x^q, \quad (21)$$

where $g_{pq} = F^{-2} \delta_{pq}$ and $A_\mu^1 = (0, a_s)$ is the magnetic and $A_\mu^2 = (F, 0)$ is the electric field ($x^\mu = (t, x^s)$).

In conclusion, we would like to emphasize that the six-dimensional conformal σ -model (11),(12) is the *minimal* one which enjoys the property of being covariant with respect to both the target space T -duality (15) and the electro-magnetic S -duality (18) or (19). Indeed, one needs *two* charges associated with each internal dimension to have covariance under T -duality, i.e., scalar duality in two dimensions, and *two* internal dimensions to embed both electric and magnetic charges, i.e., to have electro-magnetic or vector duality in four dimensions.

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²⁰Note that this theory has *two* chiral currents or affine symmetries: not only in the $y_2 = v$ direction (as in the general case of (11)) but also in the $y_1 = x_4$ direction. This is due to the special T -self-dual choice of $f = k^{-1}$, $a_s = b_s$.

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